# THE GROSS-PITAEVSKII FUNCTIONAL WITH A RANDOM BACKGROUND POTENTIAL AND CONDENSATION IN THE SINGLE PARTICLE GROUND STATE

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Abstract. For discrete and continuous Gross-Pitaevskii energy functionals with a random background potential, we study the Gross-Pitaevskii ground state. We characterize a regime of interaction coupling when the Gross-Pitaevskii ground state and the ground state of the random background Hamiltonian asymptotically coincide.

## 1. Introduction

The purpose of the present paper is to study some aspects of condensation in the ground state of the Gross-Pitaevskii energy functional with a disordered background potential. As they can be treated very similar, we consider the discrete and the continuous setting simultaneously.

**The continuous setting:**

In $\mathbb{R}^d$, consider the cube $\Lambda_L = [-L, L]^d$ of side length $2L$ and volume $|\Lambda_L| = (2L)^d$. In $\mathcal{H}_L := L^2(\Lambda_L)$, on the domain $\mathcal{D}_L := H^2(\Lambda_L)$, consider $\mathrm{H}^P_{\omega,L} = (-\Delta + V_\omega)^P_{\Lambda_L}$ the continuous self-adjoint Anderson model on $\Lambda_L$ with periodic boundary conditions. We assume

- $\Delta = \sum_{j=1}^d \partial_j^2$ is the continuous Laplace operator;
- $V_\omega$ is an ergodic random potential i.e. an ergodic random field over $\mathbb{R}^d$ that satisfies

$$\forall \alpha \in \mathbb{N}^d,\ \|\|\partial^\alpha V_\omega\|_{x,\infty}\|_{\omega,\infty} < +\infty$$

  where $\|\cdot\|_{x,\infty}$ (resp. $\|\cdot\|_{\omega,\infty}$) denotes the supremum norm in $x$ (resp. $\omega$).

These assumptions are for example satisfied by a continuous Anderson model with a smooth compactly supported single site potential i.e. if

$$V_\omega(x) = \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma u(x-\gamma)$$

where $u \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ and $(\omega_\gamma)_{\gamma\in\Lambda_L}$ are bounded, non negative identically distributed random variables.

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**The discrete setting:**

On the finite discrete cube $\Lambda_L = [-L, L]^d \cap \mathbb{Z}^d \subset \mathbb{Z}^d$ the cube of side length $2L+1$ and volume $|\Lambda_L| = (2L+1)^d$, let $\mathrm{H}^P_{\omega,L} = (-\Delta + V_\omega)^P_{\Lambda_L}$ the discrete Anderson model on $\mathcal{D}_L = \mathcal{H}_L := \ell^2(\Lambda_L)$ with periodic boundary conditions. We assume

- $(-\Delta)^P_{\Lambda_L}$ is the discrete Laplacian;
- $V_\omega$ is a potential i.e. a diagonal matrix entries of which are are given by bounded non negative random variables, say $\omega = (\omega_\gamma)_{\gamma\in\Lambda_L}$.

For the sake of definiteness, we assume that the infimum of the (almost sure) spectrum of $H_\omega$ be 0. We define

**Definition 1** (**Gross-Pitaevskii energy functional [GPEF]**)**.** The (one-particle) Gross-Pitaevskii energy functional on the cube $\Lambda_L$ (in the discrete or in the continuous) is defined by

$$\mathcal{E}^{GP}_{\omega,L}[\varphi] = \langle \mathrm{H}^{\mathrm{P}}_{\omega,\mathrm{L}}\varphi, \varphi\rangle + \mathrm{U}\|\varphi\|_4^4 \tag{1}$$

for $\varphi \in \mathcal{D}_L$ and U is a positive coupling constant.

For applications, it is natural that this coupling constant is related to $|\Lambda_L|$. We refer to the discussion following Theorem 3 for details. One proves

**Proposition 2.** *For any $\omega \in \Omega$ and $L \geq 1$, there exists a ground state $\varphi^{\mathrm{GP}}$ i.e. a vector $\varphi^{\mathrm{GP}} \in \mathcal{D}_L$ such that $\|\varphi^{\mathrm{GP}}\|_2 = 1$ minimizing the Gross-Pitaevskii energy functional, i.e.*

$$\mathrm{E}^{\mathrm{GP}}_{\omega,L} = \mathcal{E}^{GP}_{\omega,L}[\varphi^{\mathrm{GP}}] = \min_{\substack{\varphi\in\mathcal{D}_L\\ \|\varphi\|_2=1}} \mathcal{E}^{GP}_{\omega,L}[\varphi]. \tag{2}$$

*The ground state $\varphi^{\mathrm{GP}}$ can be chosen positive; it is unique up to a change of phase. $\mathrm{E}^{\mathrm{GP}}_{\omega,L}$ denotes the ground state energy of the discrete Gross-Pitaevskii functional.*

The proof in the continuous case is given in [26]; the proof in the discrete case is similar.

Let $H^N_{\omega,L}$ and $H^D_{\omega,L}$ respectively denote the Neumann and Dirichlet restrictions of $H_\omega$ to $\Lambda_L$. Our main assumptions on the random model are:

**(H0) Decorrelation estimate:** the model satisfies a finite range decorrelation estimate i.e. there exits $R > 0$ such that, for any $J \in \mathbb{N}^*$ and any sets $(D_j)_{1\leq j\leq J}$, if

$$\inf_{j\neq j'} \mathrm{d}ist(D_j, D_{j'}) \geq R,$$

then the restrictions of $V_\omega$ to the domains $D_j$, i.e. the functions $(V_{\omega|D_j})_{1\leq j\leq J}$, are independent random fields.
**(H1) Wegner estimate:** There exists $C > 0$ such that, for any compact interval $I$ and $\bullet \in \{P, N, D\}$,

$$\mathbb{E}[\mathrm{tr}\,(\mathbf{1}_I(H^\bullet_{\omega,L}))] \leq C|I|L^d;$$

**(H2) Minami estimate:** There exists $C > 0$ such that, for $I$ a compact interval and $\bullet \in \{P, N, D\}$,

$$\mathbb{P}[\{H^\bullet_{\omega,L} \text{ has at least two eigenvalues in } I\}] \leq C(|I|L^d)^2;$$

**(H3) Lifshitz type estimate near energy 0:** There exist constants $C > c > 0$ such that, for $L \geq 1$ and any parallelepiped $P_L = I_1 \times \cdots \times I_d$ where the intervals $(I_j)_{1\leq j\leq d}$ satisfy $L/2 \leq |I_j| \leq 2L$, one has

$$ce^{-L^d/c} \leq \mathbb{P}[\{H^D_{\omega|P_L} \text{ has at least one eigenvalue in } [0, L^{-2}]\}],$$

$$\mathbb{P}[\{H^N_{\omega|P_L} \text{ has at least one eigenvalue in } [0, L^{-2}]\}] \leq Ce^{-L^d/C}$$

where $H^D_{\omega|P_L}$ (resp. $H^D_{\omega|P_L}$) is the Dirichlet (resp. Neumann restriction) of $H_\omega$ to $P_L$.

Let us now discuss the validity of these assumptions.
The decorrelation assumption (H0) is satisfied for the discrete Anderson model described above if the random variables $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ are i.i.d. (H0) clearly allows some correlation between the random variables. For the continuous Anderson model, it is satisfied if the single site potential has compact support and the random variables are i.i.d.
Under the assumption that the random variables are i.i.d and that their distribution is regular, it is well known that the Wegner estimate (H1) holds at all energies for both the discrete and continuous Anderson model (see e.g. [17, 37, 7]). The Minami estimate (H2) is known to hold at all energies under similar regularity assumptions for the discrete Anderson model (see e.g. [28, 3, 13, 5]) and for the continuous Anderson model in the localization regime under more specific assumptions on the single site potential (see e.g. [6]).
Finally, the Lifshitz tails estimate (H3) is known to hold for both the continuous and discrete Anderson model under the sole assumption that the i.i.d. random variables be non degenerate, non negative and 0 is in their essential range (see e.g. [16, 18, 15]). Though the Lifshitz tails estimate is usually not stated for parallelepipeds but for cubes, the proof for cubes applies directly to parallelepipeds satisfying the condition stated in (H3).

The main result of the present paper is

**Theorem 3** (**Condensation in the single particle ground state**). *Assume assumptions (H0)-(H3) hold. Denote by $\varphi_0$ the single particle ground state of $\mathrm{H}^P_{\omega,L}$ (chosen to be positive for the sake of definiteness) and by $\varphi^{\mathrm{GP}}$ the Gross-Pitaevskii ground state.*
*If for $L$ large, one assumes that*

$$U = U(L) = o\left(\frac{L^{-d}}{(1+(\log L)^{d-2/d+\epsilon}) f_d(\log L)}\right)$$

*where*

$$f_d(\xi) = \begin{cases} \xi^{-1/4} & \text{if } d \leq 3, \\ \xi^{-1/d} \log \xi & \text{if } d = 4, \\ \xi^{-1/d} & \text{if } d \geq 5. \end{cases} \tag{3}$$

*and $\epsilon = 0$ in the discrete setting, resp. $\epsilon > 0$ arbitrary in the continuous case, then, there exists $0 < \eta(L) \to 0$ when $L \to +\infty$ such that*

$$\mathbb{P}[|\langle \varphi_0, \varphi^{\mathrm{GP}}\rangle - 1| \geq \eta(L)\}] \underset{L\to+\infty}{\to} 0. \tag{4}$$

The proof of Theorem 3 also yields information on the size of $\eta(L)$ and on the probability estimated in (4). Note that the assumption (H1)-(H3) can be relaxed at the expense of changing the admissible size for $U$.

To appreciate Theorem 3 maybe some comments about the physical background of the Gross-Pitaevskii model, its relationship to Bose-Einstein condensation and to known results are of interest. Motivated by recent experiments with weakly interacting Bose gases in optical lattices (see for example [4]) the fundamental objects of interest are the ground state density and energy, i.e.

$$\mathcal{E}^{\mathrm{QM}} := \min_{\substack{\Phi \in \bigotimes_s^N L^2(\Lambda_L) \\ \|\Phi\|=1}} \left\langle \Phi, \left[\sum_{i=1}^N \{-\Delta_i + V(x_i)\} + \sum_{1\leq i<j\leq N} v(|x_i - x_j|)\right] \Phi \right\rangle. \tag{5}$$

The optical lattice is modeled by the background potential $V$ as shown in Figure 1. Assuming a weak interaction limit of the interaction potential $v(x,y)$, the continuous $N$-particle Gross-Pitaevskii energy functional

$$\mathcal{E}^{\mathrm{GP}} = \min_{\substack{\varphi\in L^2(\Lambda_L) \\ \|\varphi\|_2=1}} \int_{\Lambda_L} \left(N|\nabla\varphi(x)|^2 + NV|\varphi(x)|^2 + 4N^2\pi\mu a|\varphi(x)|^4\right) dx, \tag{6}$$

is a mean field approximation of the ground state energy (5), e.g. in three dimensions one has

$$\lim_{N\to\infty} \frac{\mathcal{E}^{\mathrm{QM}}}{\mathcal{E}^{\mathrm{GP}}} = 1.$$

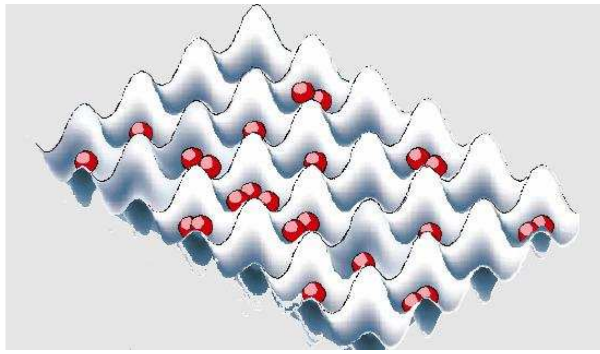

FIGURE 1. An example of a background potential modeling an optical lattice [29]

( see for example [26] ) The discrete Gross-Pitaevskii model is then a tight binding approximation of the continuous one-particle Gross-Pitaevskii functional [31, 32]

$$\mathcal{E}^{\mathrm{GP}}[\varphi] = \int_{\mathbb{R}^3} \left( |\nabla\varphi(x)|^2 + V|\varphi(x)|^2 + 4N\pi\mu a|\varphi(x)|^4 \right) dx.$$

Another way to derive the discrete Gross-Pitaevskii model starts with a discretization of (5) yielding the standard description of optical lattices using the Bose-Hubbard-Hamiltonian

$$H = -\sum_{|n-n'|=1} c_n^\dagger c_{n'} + \sum_n (\sigma V_n - \mu) n_n + \frac{1}{2} U \sum n_n^2$$

where $c_n^\dagger$, $c_n$ are bosonic creation and annihilation operators and $n_n$ gives the particle number at site $n$ (see the survey article [4] and references therein). A mean field approximation then yields the discrete Gross-Pitaevskii energy functional [23].

One motivation to study Bose gases is Bose-Einstein condensation, i.e. the phenomena that a single particle level has a macroscopic occupation ( a non-zero density in the thermodynamic limit) [26]. Introduced in [9] in the context of an ideal Bose gas, it was due to naturally arising interactions a difficult problem to realize Bose-Einstein condensation experimentally [8, 20].
As we will see, also the formal description is more elaborated. To motivate the definition of BEC for vanishing temperature we follow the continuous approach in [26]. To formalize the concept of a macroscopic occupation of a single particle state we remember the definition of the one-particle density matrix [26], i.e. the operator on $L^2(\mathbb{R}^3)$ given by the kernel

$$\gamma(x,x') = N \int \Phi^{\mathrm{QM}}(x, x_2, \ldots, x_N) \Phi^{\mathrm{QM}}(x', x_2, \ldots, x_N) \prod_{j=2}^{N} dx_j$$

with the normalized ground state wave function $\Phi^{\mathrm{QM}}$ of the many Boson Hamiltonian. BEC in the ground state is then defined that the projection operator $\gamma$

has an eigenvalue of order $N$ in the thermodynamic limit.

Remembering that for the ideal Bose gas the multi-particle ground state can be represented as a product

$$\Phi^{\mathrm{QM}}(x_1, \ldots, x_N) = \prod_{i=1}^{N} \varphi_0(x_i)$$

of the single particle ground state $\varphi_0$ the one-particle density matrix becomes

$$\gamma(x, x') = N\, \varphi_0(x)\varphi_0(x'),$$

thus the definition of BEC above is natural and can also be related to the thermodynamic formalism (see e.g. [24, 26] and references). In particular, it is of interest to consider BEC for the ideal Bose gas with a random background potential. In this case the Lifshitz tail behavior at the bottom of the spectrum makes a generalized form of Bose-Einstein condensation possible even for $d = 1$, 2 (see [24] and references cited there).

The situation in the Gross-Pitaevskii-limit is close to the situation for the ideal Bose gas [26]. The one-particle density matrix is asymptotically given by

$$\gamma(x, x') \overset{N\to\infty}{\simeq} N\, \varphi^{\mathrm{GP}}(x)\varphi^{\mathrm{GP}}(x'). \tag{7}$$

Physically the content of (7) is that all Bose particles will condensate in the GP ground state motivating the definition of complete (or 100%) BEC in [26].

The purpose of the present publication is a first step to analyze the fine structure of the Gross-Pitaevskii ground state. Under the assumption of a random background potential we want to understand how $\varphi^{\mathrm{GP}}$ is related to the eigenstates of the single particle Hamiltonian. More familiar is this problem in the following two settings.

If the Bosons are trapped by a potential tending to $\infty$, i.e. $\liminf_{|x|\to\infty} V(x) = \infty$, the spectral properties of the single particle are invariant in the thermodynamic limit, i.e. the discrete spectrum and the strictly positive distance between the first two eigenvalues. Assuming $Na \to 0$ in the continuous setting, respectively $NU \to 0$ in the context of the discrete Gross-Pitaevskii model, the interaction energy is a small perturbation of the single particle energy functional. In this situation it is natural, that in the thermodynamic limit $\varphi^{\mathrm{GP}}$ and the single particle ground state $\varphi_0$ coincide [25].

A complementary situation is given if the Bosons are confined to a cube $\Lambda_L$ with $|\Lambda_L| \to \infty$ but without a background potential. As described in [26] assuming $\rho = N/L^3$ and $g = Na/L$ in the limit $N \to \infty$ one can prove

$$\lim_{N\to\infty} \frac{1}{N}\frac{1}{L^3} \iint \gamma(x,\, y)dxdy = 1,$$

i.e. BEC in the normalized single particle ground state $\varphi_0 = L^{-d/2}\chi_{\Lambda_L}$. As explained in [26]

$$g = \frac{Na}{L} = \frac{\rho a}{1/L^2}$$

is in this context the natural interaction parameter since " in the GP limit the interaction energy per particle is of the same order of magnitude as the energy gap in the box, so that the interaction is still clearly visible".

As emphasized in the physics literature (see e.g. [4, 27]), new phenomena like fragmented BEC (Lifshitz glasses) should occur when Bosons are trapped in a random background potential. Our purpose in this publication is more modest. We want to understand the natural interaction parameter in a random media, s.t. the Gross-Pitaevskii ground state is close to the ground state of the single particle Hamiltonian as it is suggested by the situation in the ideal Bose gas. As we will see the setting of Bosons trapped in a random potential is not really comparable to the two situations described above.
Under our assumptions, near 0 which is almost sure limit of $\inf(H^P_{\omega,L})$, we are in the localized regime, i.e. one has pure point spectrum and localized eigenfunctions. In contrast to the situation with vanishing potential the eigenstates close to the bottom of the spectrum are localized in a small part of $\Lambda_L$, i.e. the interaction energy will be larger than in the case of the homogeneous Bose gas. In the random case, we determine the almost sure behavior of the ground state from information on the integrated density of states (see Lemma 5). Under our weak Lifshitz tails assumption (H3), we obtain that the ground state energy is of size $(\log L)^{-2/d}$. When $L \to +\infty$, the difference between the first two eigenvalues will tend to zero; the speed at which this happens is crucial in our analysis (see Proposition 10). In our case, we estimate that, with good probability, it must be at least of order $L^{-d}$. This difference is much smaller than the one obtained in the homogeneous Bose gas where it typically is of order $L^{-2}$. We deem that the estimate $L^{-d}$ for the spacing is not optimal in the present setting. This estimate is the correct one in the bulk of the spectrum; at the edges, the spacings should be larger. It seems that getting an optimal estimate requires a much better knowledge of the integrated density of states or, in other words, much sharper Lifshitz tails type estimates (see (H3)) and Minami type estimates that take into account the fact that we work at the edge of the spectrum (see (H2)). Combining these observations explains the interaction parameter $U = o(L^{-d} h_d^{-1}(\log L))$ that we don't believe to be optimal.

Let us now briefly outline the structure of our paper. To prove our result we need two ingredients. We need an upper bound of the interaction term, i.e. we have to estimate the $\|.\|_4$- norm of the single particle ground state $\varphi_0$. At the same time, we need a lower bound of the distance of the first two single particle eigenvalues asymptotically almost surely (a.a.s.) i.e. with a probability tending to 1 in the thermodynamic limit. Comparing these two estimates we will see that under the assumptions of Theorem 3 it is energetically favorable, that the Gross-Pitaevskii ground state and the single particle ground state coincide. This will be proven at the end of this publication.
To estimate the interaction term we will prove in Lemma 5 that almost surely in the thermodynamic limit the single particle ground state is flat, i.e.

$$\|\nabla\varphi_0\|^2 \xrightarrow{L\to\infty} 0 \qquad \text{a.a.s.}$$

This then yields an estimate of the interaction term which is the purpose of Proposition 4.
The a.a.s. lower bound of the distance of the first two single particle eigenvalues is a little bit more intricate and uses the Wegner and Minami estimates; it is related to the methods developed in [12]. In Lemma 12, we first estimate the probability that the first two eigenstates and also their localization center are close together. If the localization centers are relatively far away, one can decouple the eigenstates and treat the first two eigenvalues of each other. This is used in Lemma 13.

## 2. Estimating the interaction term

The main result of this section is an upper bound on $\mathrm{E}^{\mathrm{GP}}_{\omega,L} - \mathrm{E}^{\mathrm{P}}_{0}[\omega,\mathrm{L}]$. This quantity is non negative (see (2)) and we prove

**Proposition 4.** *There exists $C>0$, such that, for any $p\in\mathbb{N}$, one has*

$$\mathbb{P}\left[\mathrm{E}^{\mathrm{GP}}_{\omega,L} - \mathrm{E}^{\mathrm{P}}_{0}[\omega,\mathrm{L}] \leq CUf_d(\log L)\right] \geq 1-L^{-p} \tag{8}$$

*where $f_d$ is defined in (3).*

By definition, for $\varphi_0(\omega,L)$ the ground state of $H^P_{\omega,L}$, one has

$$\mathrm{E}^{\mathrm{GP}}_{\omega,L} \leq \mathcal{E}_{\omega,L}[\varphi_0(\omega,L)] = \mathrm{E}^{\mathrm{P}}_{0}[\omega,\mathrm{L}] + U\|\varphi_0(\omega,L)\|_4^4 \tag{9}$$

resp.

$$\mathrm{E}^{\mathrm{GP}}_{\omega,L} - \mathrm{E}^{\mathrm{P}}_{0}[\omega,\mathrm{L}] \leq U\|\varphi_0(\omega,L)\|_4^4.$$

To prove Proposition 4, resp. control the interaction term, we first estimate the ground state energy of the random Schrödinger operator and derive in Corollary 6 an estimate on the “flatness” of its ground state. We start with the Dirichlet and Neumann boundary cases.

**Lemma 5.** *Assume (H3) is satisfied. Let $E^P_0(\omega,L)$ be the ground state energy of $H^P_{\omega,L}$ and denote by $\varphi_0(\omega,L)$ the associated positive normalized ground state. Then, for any $p>0$, there is a constant $C>0$ such that, for $L$ sufficiently large,*

$$\mathbb{P}\left[C^{-1}(\log L)^{-2/d} \leq E^P_0(\omega,L) \leq C(\log L)^{-2/d}\right] \geq 1-L^{-p}. \tag{10}$$

As $V_\omega$ is non negative and $\varphi_0(\omega,L)$ normalized, one has $\|\nabla\varphi_0(\omega,L)\|^2 \leq E^P_0(\omega,L)$. Hence, Proposition 4 implies the following “flatness” estimate of the ground state.

**Corollary 6.** *Under the assumptions of Proposition 4, for any $p>0$, there is a constant $C>0$ such that, for $L$ sufficiently large,*

$$\mathbb{P}\left[\|\nabla\varphi_0(\omega,L)\|^2 \leq C(\log L)^{-2/d}\right] \geq 1-L^{-p}. \tag{11}$$

It is maybe interesting to note that from a Lifshitz tail type estimate (i.e. the annealed estimate), we recover the (approximate) almost sure behavior of the ground state energy of $H^N_{\omega,L}$ (i.e. the quenched estimate) (see e.g. [35]).
We note that Proposition 4 and Corollary 6 also hold if we replace the periodic ground state and ground state energy by the Neumann or Dirichlet ones.

*Proof of Lemma 5.* Fix $\ell \geq 1$. Decompose the interval $[-L, L]$ into intervals of length comprised between $\ell/2$ and $2\ell$. This yields a partition of $\Lambda_L$ in parallelepipeds i.e.

$$\Lambda_L = \bigcup_{1\leq j\leq J} P_j$$

such that

- $P_j = I^1_j \times \cdots \times I^d_j$ where the intervals $(I^k_j)_{1\leq k\leq d}$ satisfy $\ell/2 \leq |I^j_k| \leq 2\ell$
- for $j \neq j'$, $P_j \cap P_{j'} = \emptyset$,
- $J$, the number of parallelepiped, satisfies $2^{-d}(L/\ell)^d \leq J \leq 2^d(L/\ell)^d$.

In the continuous model, one can take the parallel piped to be cubes.
Denote by $\omega_{|\Lambda_L}$ the restriction of $\omega$ to $\Lambda_L$. Furthermore, let $\omega^{P,L}$ be the periodic extension of $\omega_{|\Lambda_L}$ to $\mathbb{Z}^d$ i.e. for $\beta \in \Lambda_L$ and $\gamma \in \mathbb{Z}^d$, $\omega^{P,L}_{\beta+\gamma\overline{L}} = \omega_\beta$ where $\overline{L} = 2L + 1$ in the discrete case and $2L$ in the continuous one. As $\mathrm{H}^P_\omega$ is the periodic restriction of $H_\omega$ to $\Lambda_L$, we know that $\mathrm{E}^{\mathrm{P}}_0[\omega, \mathrm{L}] = \inf \sigma(H_{\omega^{P,L}})$ where this last operator is considered as acting on the full space $\mathbb{R}^d$ or $\mathbb{Z}^d$ (see e.g. [19]).
We can now decompose $\mathbb{R}^d$ or $\mathbb{Z}^d$ into $\cup_{\gamma\in\mathbb{Z}^d} \cup^J_{j=1} (\gamma\overline{L} + P_j)$. By Dirichlet-Neumann bracketing (see e.g. [15, 17]) $H_{\omega^{P,L}}$ satisfies as an operator on $\mathbb{Z}^d$ or $\mathbb{R}^d$

$$\oplus_{\gamma\in\mathbb{Z}^d} \oplus^J_{j=1} H^N_{\omega|(\gamma\overline{L}+P_j)} \leq H_{\omega^{P,L}} \leq \oplus_{\gamma\in\mathbb{Z}^d} \oplus^J_{j=1} H^D_{\omega|(\gamma\overline{L}+P_j)}. \tag{12}$$

Define

$$\mathrm{E}^\bullet_0[\omega, \ell, \mathrm{j}] = \inf \sigma(\mathrm{H}^\bullet_{\omega|P_j}) \quad \text{for} \quad \bullet \in \{N, D\}; \tag{13}$$

here, the superscripts $D$ and $N$ refer respectively to the Dirichlet and Neumann boundary conditions. As $\omega^{P,L}$ is $\overline{L}\mathbb{Z}^d$-periodic, $\mathrm{H}^\bullet_{\omega|P_j}$ and $\mathrm{H}^\bullet_{\omega|(\gamma\overline{L}+P_j)}$ are unitarily equivalent. The bracketing (12) then yields

$$\inf_{1\leq j\leq J} \mathrm{E}^{\mathrm{N}}_0[\omega, \ell, \mathrm{j}] \leq \mathrm{E}^{\mathrm{P}}_0[\omega, \mathrm{L}] \leq \inf_{1\leq j\leq J} \mathrm{E}^{\mathrm{D}}_0[\omega, \ell, \mathrm{j}].$$

Labeling every second interval of the partition of $[-L, L]$ used to construct the partition of $\Lambda_L$, we can partition the interval $\{1, \cdots, J\}$ into $2^d$ sets, say $(\mathcal{J}_l)_{1\leq l\leq 2^d}$ such that

(1) if $l \neq l'$, $\mathcal{J}_l \cap \mathcal{J}_{l'} = \emptyset$,
(2) for $j \in \mathcal{J}_l$ and $j' \in \mathcal{J}_l$ such that $j \neq j'$, one has $\mathrm{dist}(P_j, P_{j'}) \geq \ell/2$,
(3) there exists $C > 0$ such that for $1 \leq l \leq 2^d$, $C^{-1}(L/\ell)^d \leq \#\mathcal{J}_l \leq C(L/\ell)^d$.

Assume $R$ is given by (H0). By (2) of the definition of the partition above, for any $l \geq 2R$, all the $(\mathrm{H}^{\bullet}_{\omega|P_j})_{j\in\mathcal{J}_l}$, resp. all the $(E_0^{\bullet}[\omega,\ell,j])_{j\in\mathcal{J}_l}$ (for $\bullet \in \{N, D\}$) are independent. Hence, using (13), we compute

$$\mathbb{P}[\mathrm{E}_0^{\mathrm{P}}[\omega,\mathrm{L}] > E] \leq \mathbb{P}[\inf_j \mathrm{E}_0^{\mathrm{D}}[\omega,\ell,\mathrm{j}] > E] \leq \sum_{l=1}^{2^d} \prod_{j\in\mathcal{J}_l} \mathbb{P}[\mathrm{E}_0^{\mathrm{D}}[\omega,\ell,\mathrm{j}] > E]$$
$$= \sum_{l=1}^{2^d} \prod_{j\in\mathcal{J}_l} \left(1 - \mathbb{P}[\mathrm{E}_0^{\mathrm{D}}[\omega,\ell,\mathrm{j}] \leq E]\right).$$

Pick $E = c\ell^{-2}$ where $c$ is given by assumption (H3) and

$$(k\log L - c^{-1}\log c)^{1/d} \leq \ell \leq (k\log L - c^{-1}\log c)^{1/d} + 1$$

where $k$ will be chosen below. Applying the Lifshitz estimate (H3), we obtain

$$\mathbb{P}[\mathrm{E}_0^{\mathrm{P}}[\omega,\mathrm{L}] > E] \leq \sum_{l=1}^{2^d} \left(1 - e^{-k\log L/c}\right)^{\#\mathcal{J}_l}$$
$$\leq \sum_{l=1}^{2^d} \exp\left(-\#\mathcal{J}_l\, e^{-k\log L/c}\right) \leq O(L^{-\infty})$$

if we choose $k < cd$ as $C^{-1}(L/\ell)^d \leq \#\mathcal{J}_l \leq C(L/\ell)^d$ for $1 \leq l \leq 2^d$.
Hence, we have

$$\mathbb{P}[\mathrm{E}_0^{\mathrm{N}}[\omega,\mathrm{L}] \leq E] \geq 1 - O(L^{-\infty}).$$

To estimate from below, we use again (13) to get

$$\mathbb{P}[\mathrm{E}_0^{\mathrm{P}}[\omega,\mathrm{L}] \leq E] \leq \sum_{j\in\mathcal{J}} \mathbb{P}[\mathrm{E}_0^{\mathrm{N}}[\omega,\ell,\mathrm{j}] \leq E]).$$

Pick $E = C\ell^{-2}$ where $C$ is given by assumption (H3) and

$$(k\log L - C^{-1}\log C)^{1/d} \leq \ell \leq (k\log L - C^{-1}\log C)^{1/d} + 1$$

where $k$ will be chosen below. As $\#\mathcal{J} \leq C(L/\ell)^d$, applying the Lifshitz estimate (H3), we obtain

$$\mathbb{P}[\mathrm{E}_0^{\mathrm{N}}[\omega,\mathrm{L}] \leq E] \leq C\left(\frac{L}{\ell}\right)^d e^{-k\log L/C} \leq L^{-p}$$

if we choose $k > (d+p)C$. Hence, we have

$$\mathbb{P}[\mathrm{E}_0^{\mathrm{N}}[\omega,\mathrm{L}] \geq E] \geq 1 - L^{-p}.$$

This completes the proof of Lemma 5. □

To prove estimate (8), we will use the spectral decomposition of $-\Delta_L^P$. Though the arguments in the discrete and continuous cases are quite similar, it simplifies the discussion to distinguish between the discrete and the continuous case rather than to introduce uniform notations. We start with the discrete case.

**Lemma 7.** *There exists $C>0$ such that, for $\varepsilon\in(0,1)$ and $L\in\mathbb{N}$ satisfying $L\cdot\varepsilon\geq 1$ one has for $u\in\ell^2(\mathbb{Z}^d/(2L+1)\mathbb{Z}^d)$ with $\|u\|_2=1$ and $\langle-\Delta_L^P u,u\rangle\leq\varepsilon^2$ the estimate*

$$\|u\|_4\leq Cg_d(\varepsilon)\ \text{ where } g_d(\xi)=\begin{cases}\xi^{d/4} & \text{if } d\leq 3,\\ \xi|\log\xi| & \text{if } d=4,\\ \xi & \text{if } d\geq 5.\end{cases}\tag{14}$$

*Proof.* The spectral decomposition of $-\Delta_L^P$ is given by the discrete Fourier transform that we recall now. Identify $\Lambda_L$ with the Abelian group $\mathbb{Z}^d/(2L+1)\mathbb{Z}^d$. For $u\in\mathcal{H}_L$, set

$$\hat{u}=(\hat{u}_\gamma)_{|\gamma|\leq L}\text{ where }\hat{u}_\gamma=\frac{1}{(2L+1)^{d/2}}\sum_{|\beta|\leq L}u_\beta\cdot e^{-2i\pi\gamma\beta/(2L+1)}.\tag{15}$$

Then, one checks that (see e.g. [21])

$$(-\Delta_L^P u)\hat{}=(h(\gamma)\hat{u}_\gamma)_{|\gamma|\leq L}\text{ where }h(\gamma)=2d-2\sum_{j=1}^{d}\cos\left(\frac{2\pi\,\gamma_j}{2L+1}\right).\tag{16}$$

Pick $u\in\ell^2(\mathbb{Z}^d/(2L+1)\mathbb{Z}^d)$ with $\|u\|_2=1$ and $\langle-\Delta_L^P u,u\rangle\leq\varepsilon^2$ and write $u=\sum_{k=0}^{k_\varepsilon}u_k$ where $k_\varepsilon\in\mathbb{N}$, $-\log\varepsilon\leq k_\varepsilon<-\log\varepsilon+1$ and

- $\hat{u}_0=\hat{u}\cdot\mathbf{1}_{|\gamma|<\varepsilon L}$
- for $1\leq k\leq k_\varepsilon-1$, $\hat{u}_k=\hat{u}\cdot\mathbf{1}_{e^{k-1}\varepsilon L\leq|\gamma|<e^k\varepsilon L}$
- $\hat{u}_{k_\varepsilon}=\hat{u}\cdot\mathbf{1}_{e^{k_\varepsilon}\varepsilon L\leq|\gamma|}$

where $\hat{u}$ denotes the discrete Fourier transform defined defined in (15).
Then, for $k\neq k'$, $\langle u_k,u_{k'}\rangle=0$ and, using (16), for $k\geq 1$,

$$C^{-1}\sum_{k=0}^{k_\varepsilon}(e^{k-1}\varepsilon)^2\|u_k\|_2^2\leq\sum_{k=0}^{k_\varepsilon}\langle-\Delta_L^P u_k,u_k\rangle=\langle-\Delta_L^P u,u\rangle\leq\varepsilon^2$$

i.e.

$$\sum_{k=0}^{k_\varepsilon}e^{2k}\|u_k\|_2^2\leq C.\tag{17}$$

Hence, using (15) and Hölder's inequality, we compute

$$\begin{aligned}|(u_k)_\beta|&=\frac{1}{(2L+1)^{d/2}}\left|\sum_{e^{k-1}\varepsilon L\leq|\gamma|<e^k\varepsilon L}(\hat{u}_k)_\gamma e^{-2i\pi\gamma\beta/(2L+1)}\right|\\&\leq\frac{1}{(2L+1)^{d/2}}\left(\sum_{e^{k-1}\varepsilon L\leq|\gamma|<e^k\varepsilon L}|(\hat{u}_k)_\gamma|^p\right)^{1/p}\left(\sum_{e^{k-1}\varepsilon L\leq|\gamma|<e^k\varepsilon L}1\right)^{1/q}\\&\leq\frac{1}{(2L+1)^{d/2}}\|\hat{u}_k\|_p\,(e^k\varepsilon(2L+1))^{d/q}.\end{aligned}$$

So, for $p=q=2$, one gets

$$\|u_k\|_\infty\leq\|u_k\|_2\,(e^k\varepsilon)^{d/2}.\tag{18}$$

Then, using (17), we compute

$$\|u\|_4 \leq \sum_{k=0}^{k_\varepsilon} \|u_k\|_4 \leq \sum_{k=0}^{k_\varepsilon} \sqrt{\|u_k\|_2 \|u_k\|_\infty} \leq C \sum_{k=0}^{k_\varepsilon} e^{k(d-4)/4} \varepsilon^{d/4} \leq C g_d(\varepsilon)$$

where $g_d$ is defined in (14). This completes the proof of Lemma 7. □

**Remark 8.** Lemma 7 is essentially optimal as, for $L$ sufficiently large,

- the trial function
$$u_\gamma = \begin{cases} \varepsilon \text{ if } \gamma = 0, \\ (2L+1)^{-d/2} \text{ if } \gamma \neq 0, \end{cases}$$
satisfies $1 \leq \|u\|_2 \leq 1 + \varepsilon$, $\langle -\Delta_L^P u, u \rangle \leq C\varepsilon^2$ and $\|u\|_4 \geq \varepsilon/C$;
- the trial function
$$\hat{u}_\gamma = \begin{cases} (2\varepsilon L+1)^{-d/2} \text{ if } |\gamma| \leq \varepsilon L, \\ 0 \text{ if } |\gamma| > \varepsilon L, \end{cases}$$
satisfies $\|u\|_2 = 1$, $\langle -\Delta_L^P u, u \rangle \leq C\varepsilon^2$ and $\|u\|_4 \geq \varepsilon^{d/4}/C$.

We now turn to the continuous case.

**Lemma 9.** *Fix $\eta \in (0, 1/4)$. There exists $C > 0$ such that, for $\varepsilon \in (0,1)$, $n > (d-2)\eta^{-1}+1$ and $L \in \mathbb{N}$ satisfying $L\cdot\varepsilon \geq 1$ one has for $u \in H^n(\mathbb{R}^d/(2L)\mathbb{Z}^d)$ with $\langle -\Delta_L^P u, u \rangle \leq \varepsilon^2$ the norm estimate*

$$\|u\|_4 \leq C g_{d,n}(\varepsilon) \|u\|_{H^n}^\eta \;\; \textit{where } g_{d,\eta}(\xi) = \begin{cases} \xi^{d/4} & \textit{if } d \leq 3, \\ \xi^{1-\eta} |\log \xi| & \textit{if } d = 4, \\ \xi^{1-\eta} & \textit{if } d \geq 5. \end{cases}$$

*Proof.* We now use the Fourier series transform to decompose $-\Delta_L^P$. Identify $\Lambda_L$ with the Abelian group $\mathbb{R}^d/2L\mathbb{Z}^d$. For $u \in \mathcal{H}_L$, set

$$\hat{u} = (\hat{u}_\gamma)_{\gamma \in \mathbb{Z}^d} \text{ where } \hat{u}_\gamma = \frac{1}{(2L)^{d/2}} \int_{\Lambda_L} u(\theta) \cdot e^{-\pi i \gamma \theta / L} d\theta. \tag{19}$$

Then,

$$u(\theta) = \frac{1}{(2L)^{d/2}} \sum_{\gamma \in \mathbb{Z}^d} \hat{u}_\gamma e^{\pi i \gamma \theta / L} \tag{20}$$

and

$$(-\Delta_L^P u)\hat{} = \left( \left| \frac{\pi \gamma}{L} \right|^2 \hat{u}_\gamma \right)_{|\gamma| \leq L} \text{ if } u \in \mathcal{D}_L. \tag{21}$$

Pick $u$ as in Lemma 9 and decompose it as in the proof of Lemma 7 i.e. write $u = \sum_{k=0}^{k_\varepsilon} u_k$ where $k_\varepsilon \in \mathbb{N}$, $-\log \varepsilon \leq k_\varepsilon < -\log \varepsilon + 1$ and

- $\hat{u}_0 = \hat{u} \cdot \mathbf{1}_{|\gamma| < \varepsilon L}$
- for $1 \leq k \leq k_\varepsilon - 1$, $\hat{u}_k = \hat{u} \cdot \mathbf{1}_{e^{k-1}\varepsilon L \leq |\gamma| < e^k \varepsilon L}$
- $\hat{u}_{k_\varepsilon} = \hat{u} \cdot \mathbf{1}_{e^{k_\varepsilon}\varepsilon L \leq |\gamma|}$

where $\hat{u}$ denotes the Fourier series transform defined in (19) and (20).
The control on $u_k$ for $0 \le k \le k_\varepsilon - 1$ is obtained in the same way as in the proof of Lemma 7 namely the estimate (18) holds for $0 \le k \le k_\varepsilon - 1$. The additional ingredient that we need is to obtain a control over the large frequency components.
Recall that

$$\|u\|_{H^n}^2 = \sum_{\gamma \in \mathbb{Z}^d} \left(1 + \left|\frac{\pi\gamma}{L}\right|^2\right)^{n/2} |\hat{u}_\gamma|^2$$

Fix $r > d$. For notational convenience, write $v = u_{k_\varepsilon}$ and compute

$$\begin{aligned}
|v(\theta)| &= \frac{1}{(2L)^{d/2}} \left| \sum_{e^{k_\varepsilon}\varepsilon L \le |\gamma|} (\hat{v})_\gamma e^{-i\pi\gamma\theta/L} \right| \\
&\le \frac{1}{(2L)^{d/2}} \sum_{e^{k_\varepsilon}\varepsilon L \le |\gamma|} \left[ \left|\frac{\pi\gamma}{L}\right|^{r/2} |(\hat{v})_\gamma| \right] \left|\frac{\pi\gamma}{L}\right|^{-r/2} \\
&\le \frac{1}{(2L)^{d/2}} \left( \sum_{e^{k-1}\varepsilon L \le |\gamma| < e^k \varepsilon L} \left|\frac{\pi\gamma}{L}\right|^{r} |(\hat{v})_\gamma|^2 \right)^{1/2} \left( \sum_{e^{k-1}\varepsilon L \le |\gamma|} \left|\frac{\pi\gamma}{L}\right|^{-r} \right)^{1/2} \\
&\le C \left( \sum_{e^{k-1}\varepsilon L \le |\gamma| < e^k \varepsilon L} \left|\frac{\pi\gamma}{L}\right|^{r-2/q} |(\hat{v})_\gamma|^{2/p} \cdot \left|\frac{\pi\gamma}{L}\right|^{2/q} |(\hat{v})_\gamma|^{2/q} \right)^{1/2} \\
&\le C \|v\|_{H^{rp-2p/q}}^{1/p} \cdot \langle -\Delta_L^P v, v \rangle^{1/(2q)} \\
&\le C \|v\|_{H^n}^{\eta} \cdot \epsilon^{1-\eta}
\end{aligned}$$

if $p = \eta$, $q = 1 - \eta$ and $r = (n-1)\eta + 2 > d$ as $n > (d-2)\eta^{-1} + 1$. One then completes the proof of Lemma 9 in the same way as that of Lemma 7. □

*Proof of Proposition 4.* In the discrete case Proposition 4 is a consequence of Lemma 5 and Lemma 7 with $\varepsilon = C(\log L)^{-1/d}$.

To be able to apply Lemma 9 to $\varphi_0(\omega, L)$ in the continuous case, we need to show that, for any $n > d$, $\varphi_0(\omega, L) \in H^n$ with a bounded depending only on $n$ not on $L$ or $\omega$. Therefore we use the first assumption on the random field $V_\omega$ i.e. that, for any $\alpha \in \mathbb{N}^d$, $\|\partial^\alpha V_\omega\|_{\omega,x,\infty} < +\infty$. Hence, as $\varphi_0(\omega, L)$ is an eigenvector of $-\Delta_L^P + V_\omega$, using the eigenvalue equation

$$-\Delta_L^P \varphi_0(\omega, L) = (E_0^P[\omega, L] - V_\omega)\varphi_0(\omega, L)$$

inductively, we see that

$$\| \|\varphi_0(\omega, L)\|_{H^n} \|_{\omega,\infty} < +\infty.$$

Proposition 4 in the continuous case is then a consequence of Lemma 5 and Lemma 9 with $\varepsilon = C(\log L)^{-1/d}$. This completes the proof of Proposition 4. □

## 3. The spectral gap of the random Hamiltonian

The main result of the present section is

**Proposition 10.** *Let the first two eigenvalues of* $\mathrm{H}^P_{\omega,L}$ *be denoted by* $\mathrm{E}^{\mathrm{P}}_0[\omega,\mathrm{L}] < \mathrm{E}^{\mathrm{P}}_1[\omega,\mathrm{L}]$. *Then, for* $p>0$, *there exists* $C>0$ *such that, for* $L$ *sufficiently large and* $\eta\in(0,1)$, *one has*

$$\mathbb{P}\left[\mathrm{E}^{\mathrm{P}}_1[\omega,\mathrm{L}]-\mathrm{E}^{\mathrm{P}}_0[\omega,\mathrm{L}]\leq \eta L^{-d}\right]\leq C\eta\left[1+(\log L)^{d-2/d+\epsilon}\right]+L^{-p}$$

*with* $\epsilon=0$ *in the discrete setting resp.* $\epsilon>0$ *arbitrary in the continuous case.*

In the localization regime, both the level-spacing and the localization centers spacing have been studied in e.g. [12, 14]. The main difficulty arising in the present setting is that the interval over which we need to control the spacing is of length $C(\log L)^{-2/d}$; it is large compared to the length scales dealt with in [12, 14].

Our analysis of the spectral gap relies on the description of the ground state resulting from the analysis of the Anderson model $H_\omega$ in the localized regime (see e.g. [15], [33]). Under the assumptions made above on $H_\omega$, there exists $I$ a compact interval containing 0 such that, in $I$, the assumptions of the Aizenman-Molchanov technique (see e.g. [1, 2]) or of the multi-scale analysis (see e.g.[11]) are satisfied. One proves

**Lemma 11** ([11, 22])**.** *There exists* $\alpha>0$ *such that, for any* $p>0$, *there exists* $q>0$ *such that, for any* $L\geq 1$ *and* $\xi\in(0,1)$, *there exists* $\Omega_{I,\delta,L}\subset\Omega$ *such that*

- $\mathbb{P}[\Omega_{I,\delta,L}]\geq 1-L^{-p}$,
- *for* $\omega\in\Omega_{I,\delta,L}$, *one has that, if* $\varphi_{n,\omega}$ *is a normalized eigenvector of* $\mathrm{H}_\omega|_{\Lambda_L}$ *associated to* $E_{n,\omega}\in I$, *and* $x_n(\omega)\in\Lambda_L$ *is a maximum of* $x\mapsto|\varphi_{n,\omega}(x)|$ *on* $\Lambda_L$ *then, for* $x\in\Lambda_L$, *one has,*

$$|\varphi_{n,\omega}(x)|\leq L^q\cdot\begin{cases} e^{-\alpha|x-x_n(\omega)|} & \text{in the discrete case,}\\ e^{-\alpha|x-x_n(\omega)|^\xi} & \text{in the continuous case.}\end{cases}\tag{22}$$

Note that, for a given eigenfunction, the maximum of its modulus need not be unique but two maxima can not be further apart from each other than a distance of order $\log L$. So for each eigenfunction, we can choose a maximum of its modulus that we dub center of localization for this eigenfunction.

To prove Proposition 10, we will distinguish two cases whether the localization centers associated to $\mathrm{E}^{\mathrm{P}}_0[\omega,\mathrm{L}]$ and $\mathrm{E}^{\mathrm{P}}_1[\omega,\mathrm{L}]$, say, respectively $x_0(\omega)$ and $x_1(\omega)$ are close to or far away from each other.

In Lemma 12, we show that the centers of localization being close is a very rare event as a consequence of the Minami estimate.

In Lemma 13, we estimate the probability of $\mathrm{E}_0[\omega,\mathrm{L}]$ and $\mathrm{E}_1[\omega,\mathrm{L}]$ being close to each other when $x_0(\omega)$ and $x_1(\omega)$ are far away from each other. In this case, $\mathrm{E}_0[\omega,\mathrm{L}]$ and $\mathrm{E}_1[\omega,\mathrm{L}]$ are essentially independent of each other, and the estimate is obtained using Wegner's estimate.

**Lemma 12.** *For $p > 0$, there exists $L_0 > 0$ such that, for $\lambda > 0$, $L \geq L_0$ and $\eta \in (0,1)$, one has*

$$\mathbb{P}\left[\begin{matrix} \mathrm{E}_1^{\mathrm{P}}[\omega,\mathrm{L}] - \mathrm{E}_0^{\mathrm{P}}[\omega,\mathrm{L}] \leq \eta\, L^{-d}, \\ |x_0(\omega) - x_1(\omega)| \leq \lambda(\log L)^{1/\xi} \end{matrix}\right] \leq C\eta(\log L)^{d/\xi - 2/d} + L^{-p}$$

*with $\xi = 1$ in the discrete setting resp. $\xi > 1$ arbitrary in the continuous case.*

*Proof.* Let us start with the discrete setting. Fix $p > 0$ and let $q$ be given by Lemma 11. The basic observation following from Lemma 11 is that, for $\omega \in \Omega_{I,\delta,L}$, if $x_n(\omega)$ is the localization center of $\varphi_n(\omega, L)$ and $l \leq L$, then

$$\|(H_\omega^0 - E_n^P(\omega,L))\tilde{\varphi}_n(\omega,L,l)\| + |\|\tilde{\varphi}_n(\omega,L,l)\| - 1| \leq CL^q e^{-\alpha l}. \tag{23}$$

where

- $H_\omega^0 = [H_{\omega,L}^P]_{|x_n(\omega)+\Lambda_l}$ is $H_{\omega,L}^P$ restricted to the cube $x_n(\omega) + \Lambda_l$,
- $\tilde{\varphi}_n(\omega,L,l) = \mathbf{1}_{x_n(\omega)+\Lambda_l}\varphi_n(\omega,L)$ is the eigenfunction $\varphi_n(\omega,L)$ restricted to the cube $x_n(\omega) + \Lambda_l$.

To apply the observation above we pick a covering $(C_j)_{0\leq j\leq J}$ of $\Lambda_L$ by cubes of side length of order $\log L$ i.e. $\Lambda_L \subset \bigcup_{0\leq j\leq J} C_j$. Then the number of cubes $J$ can be estimated by $J \leq CL^d(\log L)^{-d}$ and there exists $C > 0$ (depending on $\lambda$, $q$ and $\nu$) such that, if $|x_0(\omega) - x_1(\omega)| \leq \lambda \log L$ and $l \geq C\lambda \log L$, there exists a cube $C_j$ (containing $x_0(\omega)$) such that, for $L$ sufficiently large

$$\sum_{k=0}^{1}\left(\|(H_\omega^j - E_k^P(\omega,L))\tilde{\varphi}_k(\omega,L,j)\| + |\|\tilde{\varphi}_k(\omega,L,j)\| - 1|\right) \\ + |\langle\tilde{\varphi}_0(\omega,L,j), \tilde{\varphi}_1(\omega,L,j)\rangle| \leq L^{-\nu}/2$$

where we have set $q - C\lambda\alpha < -\nu$ (see (23)) and

- $H_\omega^j$ is the operator $H_\omega$ restricted to the cube $C_j + \Lambda_l$,
- $\tilde{\varphi}_k(\omega,L,j) = \mathbf{1}_{C_j+\Lambda_l}\varphi_k(\omega,L)$ for $k \in \{0,1\}$.

Let $C$ be given by Lemma 5 and define $I = [0, 2C(\log L)^{-2/d}]$. Decompose $I \subset \cup_{m=0}^{2M+1} I_m$ where

- $I_m$ are intervals of length $4\eta L^{-d}$,
- for $m \in \{0,\dots,M-1\}$, $I_{2m} \cap I_{2(m+1)} = \emptyset = I_{2m+1} \cap I_{2m+3}$,
- for $m \in \{0,\dots,M\}$, $I_{2m} \cap I_{2m+1}$ is of length $2\eta L^{-d}$.

One can choose $M \leq CL^d(\log L)^{-2/d}\eta^{-1}$. This implies that, for $L$ sufficiently large,

$$\left\{\omega;\ \begin{matrix} \mathrm{E}_1^{\mathrm{P}}[\omega,\mathrm{L}] - \mathrm{E}_0^{\mathrm{P}}[\omega,\mathrm{L}] \leq \eta\, L^{-d} \\ |x_0(\omega) - x_1(\omega)| \leq \lambda \log L \end{matrix}\right\} \subset \Omega_1 \cup \Omega_2$$

where $\Omega_1 = \Omega \setminus \Omega_{I,\delta,L}$ and

$$\Omega_2 = \bigcup_{j=1}^{J}\bigcup_{m=0}^{2M+1}\{(H_\omega)_{|C_j+\Lambda_l} \text{ has two eigenvalues in } I_m\}.$$

By Lemma 11, we know that

$$\mathbb{P}[\Omega_1] \le L^{-p}$$

Minami's estimate (H.2) and the estimate on $M$ tells us that

$$\mathbb{P}[\Omega_2] \le CL^{2d}(\log L)^{-d-2/d}\eta^{-1}(\eta L^{-d}(C\log L)^d)^2 \le C\eta(\log L)^{d-2/d}.$$

This completes the proof for the discrete setting. The proof for the continuous case is very similar. One has to replace $\mathbf{1}_{x_n(\omega)+\Lambda_l}$ by a smooth version of the characteristic function of the cube $x_n(\omega)+\Lambda_l$ (see for example [34]), resp. change the length scale $\log L$ to $(\log L)^{1/\xi}$ in the side length of the boxes where one restricts the eigenfunctions. This is necessary because of the weaker estimate in Lemma 11. This completes the proof of Lemma 12. □

We now estimate the probability of the spectral gap being small conditioned on the fact that the localization centers are far away from one another. We prove

**Lemma 13.** *For any $p > 0$, there exists $\lambda > 0$ and $C > 0$ such that, for $L$ sufficiently large and $\eta \in (0,1)$, one has*

$$\mathbb{P}\begin{bmatrix} \mathrm{E}_1^{\mathrm{P}}[\omega,\mathrm{L}] - \mathrm{E}_0^{\mathrm{P}}[\omega,\mathrm{L}] \le \eta\, L^{-d}, \\ |x_0(\omega) - x_1(\omega)| \ge \lambda(\log L)^{1/\xi} \end{bmatrix} \le C\eta + L^{-p}$$

*with $\xi = 1$ in the discrete setting, resp. $\xi > 1$ arbitrary in the continuous case.*

*Proof.* Using the same line of reasoning as in the proof of Lemma 12 we give the proof in the discrete setting.

Fix $\nu > 2d + p$ and split the interval $[0, C(\log L)^{-2/d}]$ into intervals of length $L^{-\nu}$ as in the proof of Lemma 12. By Minami's estimate, we know that, for $L$ sufficiently large

$$\mathbb{P}\left[\mathrm{E}_1^{\mathrm{P}}[\omega,\mathrm{L}] - \mathrm{E}_0^{\mathrm{P}}[\omega,\mathrm{L}] \le L^{-\nu}\right] \le C(\log L)^{-2/d}L^{\nu}L^{2(d-\nu)} \le L^{-p}. \tag{24}$$

So we may assume that $\mathrm{E}_1^{\mathrm{P}}[\omega,\mathrm{L}] - \mathrm{E}_0^{\mathrm{P}}[\omega,\mathrm{L}] \ge L^{-\nu}$.

As in the proof of Lemma 12, pick a covering of $\Lambda_L$ by cubes, say $(C_j)_{0\le j\le J}$ of side length less than $\log L$ such that $J$, the number of cubes, satisfies $J \le C(L/\log L)^d$.

Assume that $C_j$ is the cube containing $x_1(\omega)$, $\mathrm{E}_1[\omega,\mathrm{L}] - \mathrm{E}_0[\omega,\mathrm{L}] \le \eta\, L^{-d}$ and $|x_0(\omega) - x_1(\omega)| \ge \lambda \log L$. Let $\Lambda_j^c = \Lambda_L \setminus (C_j + \Lambda_{3/4\lambda\log L})$. Define the operators $(H_\omega)_{|\Lambda_j^c}$, resp. $(H_\omega)_{|C_j+\Lambda_{\lambda\log L/4}}$ to be the restriction of $H^P_{\omega,L}$ to $\Lambda_j^c$, resp. $C_j + \Lambda_{\lambda\log L/4}$, with Dirichlet boundary conditions. If $\lambda \ge 8$ and $L$ is large enough, we know that

- $\mathrm{dist}(x_0(\omega), \partial\Lambda_j^c) \ge \lambda\log L - 3/4\lambda\log L - \log L \ge \lambda\log L/8$,
- $\mathrm{dist}(x_1(\omega), \partial(C_j + \Lambda_{\lambda\log L/4})) \ge \lambda\log L/4$
- $\mathrm{dist}(\Lambda_j^c, C_j + \Lambda_{\lambda\log L/4}) \ge \lambda\log L/2 \ge R$

with $R > 0$ as in the decorrelation assumption (H0). Hence, for $\lambda$ sufficiently large, using the estimate (23) for the operators $(H_\omega)_{|\Lambda_j^c}$ and $(H_\omega)_{|C_j+\Lambda_{\lambda\log L/4}}$, we know that:

- The operator $(H_\omega)_{|C_j+\Lambda_{\lambda \log L/4}}$ admits an eigenvalue, say $\tilde{E}_1(\omega)$, that satisfies $|\tilde{E}_1(\omega) - E_1^P(\omega)| \leq L^{-2\nu}$;
- The operator $(H_\omega)_{|\Lambda_j^c}$ admits an eigenvalue, say $\tilde{E}_0(\omega)$, that satisfies $|\tilde{E}_0(\omega) - E_0(\omega)| \leq L^{-2\nu}$. Moreover, as $(H_\omega)_{|\Lambda_L^c}$ is the Dirichlet restriction of $H^P_{\omega,L}$, its eigenvalues are larger than those of $H^P_{\omega,L}$. In particular, its second eigenvalue is larger than $E_1(\omega)$. Hence, up to a small loss in probability, we may assume it is larger than $E_0(\omega) + L^{-\nu}$ as we know the estimate (24). This implies that we may assume that $\tilde{E}_0(\omega)$ is the ground state of $(H_\omega)_{\Lambda_j^c}$.

So we obtain

$$\left\{\omega; \begin{array}{r} \mathrm{E}_1^{\mathrm{P}}[\omega,\mathrm{L}] - \mathrm{E}_0^{\mathrm{P}}[\omega,\mathrm{L}] \leq \eta\, L^{-d}, \\ |x_0(\omega) - x_1(\omega)| \geq \lambda \log L \end{array}\right\} \subset \Omega_1 \cup \bigcup_{1\leq j\leq J} \Omega_j$$

where $\Omega_1 = \Omega \setminus (\Omega_{I,\delta,L} \cup \{\omega;\ \mathrm{E}_1[\omega,\mathrm{L}] - \mathrm{E}_0[\omega,\mathrm{L}] \leq L^{-\nu}\}$ and

$$\Omega_j = \left\{\omega; \mathrm{dist}(\sigma((H_\omega)_{|C_j+\Lambda_{\lambda \log L/4}}), \inf \sigma((H_\omega)_{|\Lambda_j^c})) \leq \eta L^{-d} + L^{-\nu}\right\}$$

As $(H_\omega)_{|C_j+\Lambda_{\lambda \log L/4}}$ and $(H_\omega)_{|\Lambda_j^c}$ are independent of each other, we estimate the probability of $\Omega_j$ using Wegner's estimate to obtain

$$\mathbb{P}[\Omega_j] \leq C(\eta L^{-d} + L^{-\nu})(\log L)^d.$$

Hence, one obtains

$$\begin{aligned} \mathbb{P}\left[\begin{array}{l} \mathrm{E}_1[\omega,\mathrm{L}] - \mathrm{E}_0[\omega,\mathrm{L}] \leq \eta\, L^{-d}, \\ |x_0(\omega) - x_1(\omega)| \geq \lambda \log L \end{array}\right] &\leq C(\eta L^{-d} + L^{-\nu})(\log L)^d \frac{L^d}{(\log L)^d} + 2L^{-p} \\ &\leq C(\eta + L^{-p}) \end{aligned}$$

if $\nu > d + p$.

This completes the proof in the discrete setting. To prove Lemma 13 for the continuous case, one does the same modifications as in the proof of Lemma 12 in the continuous setting. □

Setting $\varepsilon = d(1/\xi - 1)$, Proposition 10 then follows from Lemma 12 and Lemma 13.

## 4. Proof of Theorem 3

Defining $\pi_0 = |\varphi_0\rangle\langle\varphi_0|$ and applying the definition of the ground state, we can estimate

$$\begin{aligned} &\mathrm{E}_1^{\mathrm{P}}[\omega,\mathrm{L}] \|(1-\pi_0)\varphi^{\mathrm{GP}}\| + \mathrm{E}_0^{\mathrm{P}}[\omega,\mathrm{L}] \|\pi_0\varphi^{\mathrm{GP}}\| \\ &\qquad \leq \mathrm{E}^{\mathrm{GP}}_{\omega,L} \|(1-\pi_0)\varphi^{\mathrm{GP}}\| + \mathrm{E}^{\mathrm{GP}}_{\omega,L} \|\pi_0\varphi^{\mathrm{GP}}\|, \end{aligned}$$

respectively

$$\left(\mathrm{E}_1^{\mathrm{P}}[\omega,\mathrm{L}] - \mathrm{E}^{\mathrm{GP}}_{\omega,L}\right) \|(1-\pi_0)\varphi^{\mathrm{GP}}\| \leq \left(\mathrm{E}^{\mathrm{GP}}_{\omega,L} - \mathrm{E}_0^{\mathrm{P}}[\omega,\mathrm{L}]\right) \|\pi_0\varphi^{\mathrm{GP}}\|.$$

As a consequence of Proposition 4 and Proposition 10, we know with a probability larger than $1-(C\eta+L^{-p})$ that, for $\eta\in(0,1)$ and $f_d$ defined in (3) the estimates

$$\mathrm{E}_1^{\mathrm{P}}[\omega,\mathrm{L}]-\mathrm{E}_{\omega,L}^{\mathrm{GP}}\geq \mathrm{E}_1^{\mathrm{P}}[\omega,\mathrm{L}]-\mathrm{E}_0^{\mathrm{P}}[\omega,\mathrm{L}]\geq \eta L^{-d}[1+(\log L)^{d-2/d+\epsilon}]^{-1}$$

and

$$\mathrm{E}_{\omega,L}^{\mathrm{GP}}-\mathrm{E}_0^{\mathrm{P}}[\omega,\mathrm{L}]\leq CUf_d(\log L)$$

are satisfied. We obtain

$$\|(1-\pi_0)\varphi^{\mathrm{GP}}\|\leq CUf_d(\log L)\eta^{-1}L^d[1+(\log L)^{d-2/d+\epsilon}]\ \|\pi_0\varphi^{\mathrm{GP}}\|$$

and

$$\begin{aligned}|\langle\varphi_0,\varphi^{\mathrm{GP}}\rangle|^2&=\|\pi_0\varphi^{\mathrm{GP}}\|^2=1-\|(1-\pi_0)\varphi^{\mathrm{GP}}\|^2\\&\geq 1-\Big[CUf_d(\log L)\eta^{-1}L^d[1+(\log L)^{d-2/d+\epsilon}]\Big]^2.\end{aligned}$$

Applying the assumption concerning the coupling constant $U$ i.e.

$$U=U(L)=o\left(L^{-d}[1+(\log L)^{d-2/d+\epsilon}]^{-1}[f_d(\log L)]^{-1}\right)$$

and setting

$$\eta=\eta(L)=\sqrt{|U(L)L^d[1+(\log L)^{d-2/d+\epsilon}]f_d(\log L)|}$$

we get that, $\eta(L)\to 0$ when $L\to+\infty$ and for some $C>0$,

$$\mathbb{P}(\{\omega;||\langle\varphi_0,\varphi^{\mathrm{GP}}\rangle|-1|\geq C\eta(L)\})\leq C(\eta(L)+L^{-p}).$$

This completes the proof of Theorem 3.

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